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On a formula of Coll–Gerstenhaber–Giaquinto

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Abstract

Given a bialgebra B we present a unifying approach to deformations of associative algebras A with a left B-module algebra structure. Special deformations of the comultiplication of B yield universal deformation formulas, i.e. define deformations of the multiplicative structure for *all* B-module algebras A. This allows to derive known formulas of Moyal–Vey (1949) and Coll–Gerstenhaber–Giaquinto (1989) from a more general point of view. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let K be a ring containing the field $\mathbf{Q}$ of rational numbers, $K' = K[[h]]$ be the algebra of formal series on h and $(A; \mu_A, 1_A)$ a K-algebra with unit. This algebra structure extends in a natural way by K'-linearity to the algebra $A' := A[[h]]$ of power series in h with coefficients in A that we will denote by some abuse of notation also by μ_A. The aim of this paper is to study deformations of this structure.

Definition 1. A (formal) *deformation* of the K-algebra A is an algebra structure $A_h = (A', \mu_h, 1_A)$ on A' with

$$\mu_h := \mu_A + \sum_{k=1}^{\infty} h^k \varphi_k : A' \otimes A' \to A'.$$

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For μ_h to be associative in first order on h, φ_1 must fulfill the property

$$\varphi_1(a_1, a_2, a_3) + \varphi_1(a_1, a_2), \qquad a_3 = \varphi_1(a_1, a_2 a_3) + a_1 \varphi_1(a_2, a_3)$$

for $a_1, a_2, a_3 \in A$, i.e. has to be a 2-cocycle in the Hochschild complex of A. Such a 2-cocycle φ_1 is called an *infinitesimal* of the deformation. We restrict ourselves to the case when the 2-cochains φ_k have the form $\varphi_k = \mu_A \circ P^{(k)}$, where $P^{(k)} : A \otimes A \to A \otimes A$ are K-linear maps that are induced from an action of a coalgebra B on A. Given a 2-cocycle $S := P^{(1)}$ of B we try to define $P^{(k)}$ for $k \geq 2$ so that μ_h is associative.

In practical applications such a 2-cocycle often appears as the product of 1-cocycles $S = D \otimes E$, where D, E are elements of a certain Lie algebra $\mathcal{G}$ acting by derivations on A. This generalizes the action of $\mathcal{G}$ as left invariant vector fields on the algebra of smooth functions $C^\infty(G)$, where G is the simply connected Lie group associated with $\mathcal{G}$.

There are two famous results that describe prolongations of such 2-cocycles to associative multiplications on A_h:

Theorem 2 (Moyal–Vey [5,9]). *If the Abelian Lie algebra $\mathcal{G}$ acts on a K-algebra A by derivations, then for any element $S \in \mathcal{G} \otimes \mathcal{G}$ the composition $\mu_A \circ S$ is a 2-cocycle and the multiplication*

$$\mu_h = \mu_A \circ \mathrm{e}^{hS}$$

is associative.

Theorem 3 (Coll et al. [2]). *If the 2-dimensional Lie algebra $\mathcal{G}$ with generators E, D and commutator relation $[E, D] = E$ acts on the K-algebra A by derivations, then for $S = E \otimes D$ the composition $\mu_A \circ S$ is a 2-cocycle and the multiplication*

$$\mu_h = \mu_A \circ (1 + hE \otimes 1)^{1 \otimes D}.$$

is associative.

Both theorems were first proved by direct calculations. For Moyal–Vey's theorem these computations are straightforward and use only the Leibniz rule, since D and E commute. The second result is less elementary. We will refer to this example as Gerstenhaber's.

Such derivations may be extended to a (left) B-module structure on the algebra A in the sense of [8, 1.6.1] with $B = U(\mathcal{G})$, the universal enveloping (bi)algebra of $\mathcal{G}$. This more general point of view will be discussed below.

More precisely, we leave the setting of universal enveloping algebras and define, for a bialgebra B, conditions on an element $P \in (B \otimes B)[[h]]$ such that for *any* B-module algebra A the composition $\mu_h = \mu_A \circ (P \triangleright)$ yields a deformation of A, where $\triangleright$ is induced by the B-action on A. Thus we construct universal deformation formulas in the spirit of [6].

This approach allows to derive the above results as partial cases of a more general principle to construct algebra deformations. It turns out that in this frame deformations of μ_A are close related to deformations of the comultiplication of B thus leaving the class of universal enveloping algebras.

Different aspects of such a theory are demonstrated on Gerstenhaber's example. It turns out that the tight connection between the deformation of the algebra structure of $A[[h]]$, the comultiplication of $B[[h]]$, and the adjustment of the 2-cocycle S described in the main theorem (Theorem 10) allows to construct deformation formulas step by step, increasing the order of h taken into account.

Some of the ideas were already considered in [11,12].

2. Bialgebras and B-module algebras

Let $(B; \mu_B, 1_B; \Delta_B, \epsilon_B)$ be a bialgebra with multiplication μ_B, unit 1_B, comultiplication Δ_B, and counit ϵ_B as defined, for example, in [8]. We often omit the index B and use the standard notion where an integer index of an operator, acting on a tensor product, denotes the tensor cofactor, on which the operator acts. For $b \in B$ we use the Sweedler notation $\Delta(b) = \sum b_{(1)} \otimes b_{(2)}$ and $\Delta_1\Delta(b) = \Delta_2\Delta(b) = \sum b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$ if we need to exploit their special structure as elements of $B \otimes B$, resp. $B \otimes B \otimes B$.

For a K-coalgebra C there is a notion of cohomology groups $H^n(K, C)$ as explained e.g. in [7, Ch. 18.5]. For a k-cocycle $S \in C^{\otimes k}$ the coboundary formula is defined as

$$\delta S = 1 \otimes S + \sum_{i=1}^{k}(-1)^i \Delta_i S + (-1)^{k+1} S \otimes 1.$$

Especially, a 1-cocycle $X \in C$ fulfills the condition $\Delta(X) = X_1 + X_2$. For a 2-cocycle $S \in C \otimes_K C$ we get $\Delta_2(S) + S_{23} = \Delta_1(S) + S_{12}$.

Definition 4. For a given bialgebra B a (left) *B-module algebra* A in the sense of [8, 1.6.1] is an algebra $(A, \mu_A, 1)$ with a left B-module action $\rhd$ such that μ_A and Δ_B satisfy additionally the compatibility conditions

$$\forall b \in B,\ \forall a_1, a_2 \in A: \quad b \rhd (a_1 a_2) = \sum (b_{(1)} \rhd a_1)\,(b_{(2)} \rhd a_2), \tag{1}$$

$$\forall b \in B: \quad b \rhd 1_A = \epsilon(b) \cdot 1_A. \tag{2}$$

This definition generalizes to bialgebras the concept of actions of universal enveloping algebras induced by Lie algebras of derivations. Indeed, given an algebra A and a Lie algebra $\mathcal{G}$ acting on A, the universal enveloping algebra $B = U(\mathcal{G})$ has a natural bialgebra structure with comultiplication Δ defined by $\Delta(X) = X \otimes 1 + 1 \otimes X$ for $X \in \mathcal{G}$ and A is a B-module algebra iff for $X \in \mathcal{G}$ and $a_1, a_2 \in A$

$$X \rhd (a_1 \cdot a_2) = (X \rhd a_1), \qquad a_2 + a_1(X \rhd a_2),$$

i.e. X acts as derivation on A.

Below we will only exploit condition (1), hence most of our conclusions remain valid for bialgebras without counit. For such an algebra $B = (B, \mu_B, 1_B)$ with (compatible) comultiplication Δ_B we define a B-module action on A, satisfying (1) to be *admissible*.

If no confusion arises, the $\triangleright$ sign will be omitted and $b \in B$ will be identified with its action $b \triangleright \in \mathrm{End}_K(A)$. Hence condition (1) may be reformulated as

$$b \circ \mu_A = \mu_A \circ \Delta_B(b). \tag{3}$$

Note that an action of a bialgebra B on a K-algebra A is uniquely defined by the action of the generators of B on the generators of A.

B-module algebras are quite ubiquitous as explained in [8, 1.6.]. Let us add some more examples:

1. The left action of $B = A$ on itself is an admissible action, if we define $\Delta(a) = a \otimes 1$ for $a \in B$. Analogously the right action of $B = A^{\mathrm{op}}$ on A is an admissible action w.r.t. $\Delta(a) = 1 \otimes a$.
   This may be extended to an admissible action of the enveloping algebra $A^{\mathrm{e}} := A \otimes_K A^{\mathrm{op}}$ on A, where the comultiplication is given by the rule $\Delta(x \otimes y) = (x \otimes 1) \otimes (1 \otimes y)$. If $A^{\mathrm{e}} = \mathrm{End}_K(A)$, e.g. for a matrix algebra $M_n(K)$, this construction allows to define an admissible action of the whole algebra of endomorphisms $\mathrm{End}_K(A)$ on A.
2. The natural action of the bialgebra $B = K[\partial/\partial x_1, \ldots, \partial/\partial x_n]$ defines a B-module algebra structure on $A = K[x_1, \ldots, x_n]$, since B is the universal enveloping algebra of an Abelian Lie algebra acting on A by derivations.
3. This action may be extended by left action of A on itself to an admissible action of the Weyl algebra $W = A \otimes_K B$ on A, where the multiplication on W is induced by the commutation rules
   $$\frac{\partial}{\partial x_i} \cdot x_j = \delta_{ij} + x_j \cdot \frac{\partial}{\partial x_i},$$
   and the comultiplication by the corresponding rules on A and B
   $$\Delta(x_i) = x_i \otimes 1 \quad \text{and} \quad \Delta\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i} \otimes 1 + 1 \otimes \frac{\partial}{\partial x_i}.$$
   This may easily be generalized to arbitrary Lie algebras $\mathcal{G}$ acting on A by derivations.
4. The same is true for any bialgebra B and B-module algebra A, if the corresponding multiplication on $W = A \otimes_K B$ is induced by the commutation rules
   $$b \cdot a = \sum (b_{(1)} \triangleright a) \cdot b_{(2)},$$
   and the comultiplication again by the corresponding rules on A and B. Here and below $a \in A$ and $b \in B$ are identified with their images in W under the embeddings $A \to A \otimes 1 \subset W$ and $B \to 1 \otimes B \subset W$. This is the well-known *left cross product* $A \rtimes B$, see [8, 1.6.6].
5. This may be generalized once more: There is also a natural multiplication and comultiplication on the K-module $W := A^{\mathrm{e}} \otimes B$ extending those of A^e and B, such that W acts admissible on A. As above we have only to define the product $b \cdot (x \otimes y)$ for $b \in B$, $x \otimes y \in A^{\mathrm{e}}$. As easily seen the correct rule is
   $$b \cdot (x \otimes y) = \sum \big((b_{(1)} \triangleright x) \otimes (b_{(3)} \triangleright y)\big) \cdot b_{(2)}.$$
   Note that these Weyl algebras do not admit a counit in general.

6. For a bialgebra B its dual B^* has a natural B-module algebra structure

$$\langle u \triangleright b^*, v\rangle := \langle b^*, v \cdot u\rangle \quad \text{for } b^* \in B^*, \quad u, v \in B,$$

if we define the multiplication on B^* by the rule

$$\langle a^* \cdot b^*, w\rangle := \langle a^* \otimes b^*, \Delta(w)\rangle \quad \text{for } a^*, b^* \in B^*, \quad w \in B.$$

Here $\langle b^*, w\rangle$ denotes the canonical pairing between B^* and B. The associativity of μ_{B^*} is a consequence of the coassociativity of Δ.

3. Deformations of B-module algebras

The main idea of this section is the observation that for both formulas considered in the introduction the deformed multiplication has the form $\mu_h = \mu_A \circ P$ for a certain element $P \in (B \otimes B)[[h]]$ over the bialgebra $B = U(\mathcal{G})$. Hence as for μ_{B^*} in the above example one can try to exploit the coassociativity of Δ_B to prove associativity of μ_h. In the spirit of universal deformation formulas we will ask for a condition on P such that μ_h becomes associative at once for *all* B-module algebras A.

Assume we are given a bialgebra B and a B-module algebra A as in Section 2. The scalar extension $K \to K' = K[[h]]$ defines a natural bialgebra structure on $B' = B[[h]]$, by some abuse of notation denoted $(B'; \mu_B, 1_B; \Delta_B, \epsilon_B)$, and a B'-module structure on $(A' = A[[h]], \mu_A, 1_A)$. Below we consider the question, how deformations of the algebra structure on A are related to the bialgebra B.

Let us consider the condition that must be fulfilled by an element

$$P = 1 + \sum_{i=1}^{\infty} h^i P^{(i)} \in B' \otimes_{K'} B' = (B \otimes_K B)[[h]]$$

for $\mu_h = \mu_A \circ P$ to be associative:

$$0 = \mu_h \circ (\mu_{h,12} - \mu_{h,23}) = \mu \circ P \circ (\mu_{12} \circ P_{12} - \mu_{23} \circ P_{23}).$$

Since B acts admissible we get by (3)

$$P \circ \mu_{12} = \mu_{12} \circ \Delta_1(P), \qquad P \circ \mu_{23} = \mu_{23} \circ \Delta_2(P),$$

and altogether

$$0 = \mu \circ \mu_{12} \circ (\Delta_1(P)P_{12} - \Delta_2(P)P_{23}).$$

Hence

$$\Delta_1(P)P_{12} - \Delta_2(P)P_{23} = 0 \tag{4}$$

is a sufficient condition for P to make μ_h associative for *any* B-module algebra A.

A condition similar to (4) was first considered by Drinfel'd in [3], who showed that for $B = U(\mathrm{gl}_n)$ it is essentially equivalent to the condition that $R = P_{21}^{-1} P_{12}$ fulfills the quantum Yang–Baxter equation. Later on it turned out that there is a close connection to twists of the comultiplication of bialgebras as defined e.g. in [1, 4.2.14]. Since universal deformation formulas in the above sense are essentially consequences of certain coassociativity conditions on B one may not wonder that these twists play a crucial role in our considerations, too. We will come back to them below.

For the moment let us first note that (4) yields already a one-line proof of the following generalization of the Moyal–Vey formula.

Theorem 5. *If A is a B-module algebra over the* commutative *bialgebra B then for any 2-cocycle $S \in B \otimes B$ the multiplication*

$$\mu_h = \mu_A \circ e^{hS}$$

is associative.

Proof. Indeed, for $P = e^{hS}$ condition (4) is equivalent to

$$e^{h\Delta_1(S)} \circ e^{hS_{12}} = e^{h\Delta_2(S)} \circ e^{hS_{23}}, \tag{5}$$

and finally to $\Delta_1(S) + S_{12} = \Delta_2(S) + S_{23}$. □

As an example let us consider the commutative bialgebra B with the free generators $E_i, D^i, L_i^j,\ i, j = 1, \ldots, n$ and the comultiplication that using the matrix notation

$$\mathbf{E} = (E_1 \quad E_2 \quad \cdots \quad E_n), \qquad \mathbf{D} = \begin{pmatrix} D^1 \\ D^2 \\ \vdots \\ D^n, \end{pmatrix} \qquad \mathbf{L} = (L_i^j)$$

may be written in the following form:

$$\Delta(\mathbf{E}) = \mathbf{E}_1\mathbf{L}_2 + \mathbf{E}_2, \qquad \Delta(\mathbf{D}) = \mathbf{D}_1 + \mathbf{L}_1\mathbf{D}_2,\ \Delta(\mathbf{L}) = \mathbf{L}_1\mathbf{L}_2.$$

Then the 2-cochain $S = \mathbf{E}_1\mathbf{D}_2 = \sum_{i=1}^n E_i \otimes D^i$ is a cocycle and the power series $P = e^{hS}$ satisfies Eq. (4).

This yields an explicit formula for a deformation of any B-module algebra A that does not fit into the frame of Theorem 2.

Note that the proof of the above theorem may be generalized to non-commutative bialgebras if only the exponents in (5) mutually commute:

Theorem 6. *Let S be a 2-cocycle of a (not necessarily commutative) bialgebra B and $[\Delta_1(S), S_{12}] = [\Delta_2(S), S_{23}] = 0$. Then $P = \exp(hS)$ satisfies* (4).

4. A differential equation

The solution $P = e^{hS}$ of (4) described in Theorem 5 is expressed as an exponential function. Since $f(x,h) = e^{hx}$ is the solution of the differential equation $\partial f/\partial h = x \cdot f$ with initial condition $f(x,0) = 1$ the "infinitesimal"

$$S_h := P^{-1}\frac{\partial P}{\partial h} \in (B \otimes B)[[h]] \tag{6}$$

of P also may play a crucial role for other applications. Note that the power series P is uniquely defined by S_h but their connection may be more difficult to describe than in the commutative case and for constant S_h as in Theorem 5. Since $S_h|_{h=0} = P^{(1)}$ coincides with the element $S \in B \otimes B$ defined in Section 1, S_h is a deformation of S (in a sense to be specified).

Under certain additional assumptions condition (4) may be reformulated as a condition on S_h. For example, if $B_h = (B'; \mu_B, 1_B; \Delta_h, \epsilon_B)$ is a commutative bialgebra structure on B' with a comultiplication, not induced from B, and S_h a non-constant 2-cocycle of B_h, we get as above, that $P = \exp(\int S_h\, \mathrm{d}h)$ satisfies (4) for $\Delta = \Delta_h$, and thus yields a deformation of μ_A for any B_h-module algebra $(A', \mu_A, 1_A)$.

Theorem 7. *If A' is a B_h-module algebra over the commutative bialgebra $B_h = (B'; \mu_B, 1_B; \Delta_h, \epsilon_B)$ then for any (not necessarily constant) 2-cocycle $S_h \in B \otimes B[[h]]$ of B_h the multiplication*

$$\mu_h = \mu_A \circ \exp\left(\int_0^h S_h\, \mathrm{d}h\right),$$

on A' is associative.

As an example consider the *commutative* bialgebra $B_h = K'[E, D]$ with comultiplication induced by

$$\Delta_h(E) = E_1 + E_2 + h\, E_1\, E_2, \qquad \Delta_h(D) = D_1 + (1 + h\, E_1)^{-1} \cdot D_2.$$

Coassociativity can easily be proved using the multiplicative matrix

$$\begin{pmatrix} (1 + h\, E)^{-1} & D \\ 0 & 1 \end{pmatrix}.$$

The 2-cocycle

$$S_h = \frac{E}{1 + h\, E} \otimes D = \frac{E_1}{1 + h\, E_1} \cdot D_2$$

yields after integration $P = (1+h\, E_1)^{D_2}$, i.e. Gerstenhaber's formula, but for a *commutative* bialgebra and a *deformed* B'-module action, where D and E act as derivations only up to first order.

5. A first proof of Gerstenhaber's formula

With some more effort we also may prove Gerstenhaber's formula in its original setting. Denote $\psi(x, y) = (1 + hx)^y$ so that $P = (1 + hE_1)^{D_2}$ from Theorem 3 may be rewritten as $P = \psi(E_1, D_2)$. By (4) we only have to show that

$$\psi(E_1 + E_2, D_3)\,\psi(E_1, D_2) = \psi(E_1, D_2 + D_3)\,\psi(E_2, D_3). \tag{7}$$

To see this lets first collect several helpful identities:

Lemma 8. *For $f, g \in K[x][[h]]$ and D, E with $[E, D] = E$ we get*
(1) $E^n f(D) = f(D+n)E^n$,
(2) $[D, f(E)] = -x(\partial/\partial x) f(x) \mid_{x=E}$,
(3) $f(E)D = (D + E(\partial/\partial E) \ln f(E)) \cdot f(E)$,
(4) $f(E)g(D) = g(D + E(\partial/\partial E) \ln f(E)) \cdot f(E)$ *(note that $g(D + E(\partial/\partial E) \ln f(E))$ is a function with non commuting arguments!),*
(5) *Applying the definition*

$$\binom{x}{k} := \frac{x(x-1)\cdots(x-k+1)}{k!}$$

of binomial coefficients to $x = D$ we get

$$e^{hED} = \sum_{k=0}^{\infty} h^k E^k \binom{D}{k} = (1 + hE)^D.$$

(6) $f(E)e^{\alpha D} = e^{\alpha D} f(e^{\alpha} E)$ *and* $e^{\alpha D} f(E) = f(E/e^{\alpha})e^{\alpha D}$.
In particular
(7) $(1 + hx)^D f(E) = f(E/1 + hx)(1 + hx)^D$.

Proof. These formulas may be proved immediately by straightforward computations. Conditions (1)–(5) follow almost directly from the commutation rule $[E, D] = E$ and linearity. To prove condition (6) we obtain from condition (1). for $f = \sum a_k x^k$

$$\begin{aligned} f(E)e^{\alpha D} &= \sum_{k=0}^{\infty} a_k E^k e^{\alpha D} = \sum_{k=0}^{\infty} a_k e^{\alpha(D+k)} E^k = \sum_{k=0}^{\infty} a_k e^{\alpha D} (e^{\alpha k} E^k) \\ &= e^{\alpha D} \sum_{k=0}^{\infty} a_k (e^{\alpha} E)^k = e^{\alpha D} f(e^{\alpha} E). \qquad \square \end{aligned}$$

There is a more rigid result than Theorem 3:

Theorem 9. *A power series $f(x, y) \in K[x, y][[h]]$ with $f(0, y) = 1$, $f_x(0, y) = h y$ satisfies (7) iff $f = \psi$, i.e.*

$$f(x, y) = (1 + hx)^y = \sum_{k=0}^{\infty} h^k x^k \binom{y}{k}.$$

Proof. Replacing in (7) the commuting variables E_1, D_3 by x resp. y and the remaining non-commuting D_2, E_2 by D, E we have to solve the equation

$$f(x+E,y)f(x,D)=f(x,D+y)f(E,y).$$

We will solve this functional equation transforming it into a differential equation for f. Take the first derivative with respect to x

$$f_x(x+E,y)f(x,D)+f(x+E,y)f_x(x,D)=f_x(x,D+y)f(E,y)$$

and set $x=0$. With $f(0,y)=1$, $f_x(0,y)=hy$ we get

$$f_x(E,y)+f(E,y)\,hD=h\,(D+y)\,f(E,y)$$

or

$$f_x(E,y)=h\,[D,f(E,y)]+hy\,f(E,y). \tag{8}$$

Lemma 8 yields

$$[D,f(E,y)]=-E\frac{\partial}{\partial E}f(E,y)=-Ef_x(E,y).$$

Substituting this expression in (8) we get an equation in E only.

$$f_x(E,y)=-hEf_x(E,y)+hyf(E,y).$$

Its integral with respect to the initial conditions yields $f(x,y)=(1+hx)^y$ and vice versa. □

6. A bialgebra deformation

Let $(H;\mu_H,1_H;\Delta_H,\epsilon_H)$ be a bialgebra and $P\in H\otimes H$ an invertible element such that

$$\Delta_1(P)P_{12}=\Delta_2(P)P_{23} \quad \text{and} \quad \epsilon_1(P)=\epsilon_2(P)=1_H.$$

Then the *twist* $H^P:=(H;\mu_H,1_H;\Delta_H^P,\epsilon_H)$ *of* H *by* P, with

$$\Delta_H^P(h)=P^{-1}\Delta_H(h)P \quad \text{for } h\in H,$$

is also a bialgebra, see [1, 4.2.13]. For a Hopf algebra H the twist has even a Hopf algebra structure. Twists of cocommutative Hopf algebras are triangular Hopf algebras with universal R-matrix $R=P_{21}^{-1}P_{12}$, see [1, 4.2.14], and hence close related to the quantum Yang–Baxter equation.

Since the first condition on P is exactly the universal associativity condition (4), such twists play also a central role in the following theorem.

Let B be a bialgebra and A a B-module algebra as defined above. Assume that $P\in 1+h(B\otimes B)[[h]]$ satisfies condition (4) and $\epsilon_1(P)=\epsilon_2(P)=1_B$. Then the twisted bialgebra $B_h = (B')^P$ may be considered as a deformation of B'. Hence we write Δ_h instead of Δ^P.

Theorem 10. *These assumptions imply:*
(i) $A_h = (A', \mu_h = \mu_A \circ P, 1_A)$ *is a* K'*-algebra, i.e.* μ_h *is associative.*
(ii) A_h *is a* B_h*-module algebra (w.r.t. the same* B'*-action).*
(iii) $S_h = P^{-1}(\partial P)/\partial h)$ *is a 2-cocycle of the coalgebra* (B_h, Δ_h) *that prolongates the 2-cocycle* $S = P^{(1)}$ *of the coalgebra* (B, Δ) *and defines* P *uniquely.*

Note that the additional condition on P forces ϵ_B to be a counit of B_h. It is automatically satisfied for graded bialgebras and may be skipped in the more general setting of admissible actions of an algebra B with compatible comultiplication.

Proof. μ_h is associative by (4).

$b \circ \mu_h = \mu_h \circ \Delta_h(b)$, i.e. $b \circ \mu_A \circ P = \mu_A \circ \Delta_B(b) \circ P$ follows immediately from (3) for B.

Since $(\partial/\partial h)P = PS_h$ the derivative of (4) yields

$$\Delta_1(PS_h)P_{12} + \Delta_1(P)P_{12}S_{h,12} = \Delta_2(PS_h)P_{23} + \Delta_2(P)P_{23}S_{h,23}.$$

Note that further

$$\Delta_1(PS_h)P_{12} = \Delta_1(P)\Delta_1(S_h)P_{12} = \Delta_1(P)P_{12}\Delta_{h,1}(S_h)$$

and also

$$\Delta_2(PS_h)P_{23} = \Delta_2(P)P_{23}\Delta_{h,2}(S_h).$$

With (4) we obtain

$$\Delta_1(P)P_{12} \cdot \delta_h(S_h) = 0.$$

Hence $\delta_h(S_h) = 0$ since the first cofactor is invertible. □

This theorem shows that our approach to algebra deformations through B-module algebras is a very natural one. It does not only allow to formulate a condition on P that implies the associativity of $\mu_h = \mu_A \circ P$ but also yields a deformation of the coalgebra structure on B in such a way that the deformation process may be iterated. It is this point where we leave the original setting of (universal enveloping algebras of) Lie algebras acting by derivations, since the deformed comultiplication rule is usually more difficult.

Let us explain these changes on Gerstenhaber's example. For $P = (1 + hE_1)^{D_2}$ we get as new comultiplication

$$\Delta_h(E) = P^{-1}\Delta_B(E)P = (1 + hE_1)^{-D_2}(E_1 + E_2)(1 + hE_1)^{D_2}.$$

Applying the rules collected in Lemma 8 we obtain

$$\Delta_h(E) = E_1 + E_2(1 + hE_1)^{-D_2+1}(1 + hE_1)^{D_2} = E_1 + (1 + hE_1)E_2$$

and in the same way

$$\Delta_h(D) = P^{-1}\Delta(D)P = (1+hE_1)^{-D_2}D_1(1+hE_1)^{D_2} + D_2$$
$$= D_1 - hD_2E_1(1+hE_1)^{-1} + D_2,$$

since

$$[(1+hE_1)^{-D_2}, D_1] = E_1\frac{\partial}{\partial E_1}(1+hE_1)^{-D_2} = -hD_2E_1(1+hE_1)^{-D_2-1}.$$

Introducing the (invertible) element $L := 1 + h\,E \in B_h$ we get

$$\Delta_h(E) = E_1 + L_1E_2, \quad \Delta_h(D) = D_1 + L_1^{-1}D_2, \quad \Delta_h(L) = L_1L_2.$$

Note that these are the same formulas for Δ_h as for the *commutative* bialgebra B_h at the end of Section 4.

Due to the last formula $\ln(L)$ is a lifting of the B-cocycle E to a B_h-cocycle. Since

$$S_h = P^{-1}\frac{\partial P}{\partial h} = L_1^{-D_2}\,E_1\,D_2\,L_1^{D_2-1} = L_1^{-1}\,E_1\,D_2$$

we get $\delta_h(D) = h\,S_h$, i.e. the B-cocycle D is not liftable. S is a bialgebra analog of a jump cocycle as defined in [6, p. 19] since $S = S_h|_{h=0}$ and $S_h = (1/h)\delta_h(D)$ is a coboundary for $h \neq 0$.

7. Another derivation of Gerstenhaber's formula

Over $K'[h^{-1}]$ the bialgebra B_h considered in Section 6 may even be generated by D and L. Its bialgebra structure is uniquely defined by the h-independent relations

$$\Delta(D) = D_1 + L_1^{-1}D_2, \quad \Delta(L) = L_1L_2, \quad [L, D] = L - 1. \tag{9}$$

It turns out that these relations already imply Gerstenhaber's formula. This suggests the following generalization.

Theorem 11. *Let $\tilde{B}$ be a K'-bialgebra and $L, D \in \tilde{B}$ such that $L - 1 \in h\tilde{B}$, hence L^{-1} exists, and relations* (9) *are fulfilled. Then the power series $P = L_1^{-D_2} = \exp(-\ln L_1 \cdot D_2)$ satisfies Eq.* (4).

Proof. For our P Eq. (4) has the form

$$(L_1L_2)^{-D_3} \cdot L_1^{-D_2} = L_1^{-D_2-L_2^{-1}D_3} \cdot L_2^{-D_3}$$

or

$$L_1^{-D_3} \cdot L_2^{-D_3} \cdot L_1^{-D_2} = L_1^{-D_2-L_2^{-1}D_3} \cdot L_2^{-D_3} \tag{10}$$

Here only L_2 and D_2 do not commute. In order to exchange the two factors $L_2^{-D_3}$ and $L_1^{-D_2}$ in the left-hand side we introduce the element $E := L - 1$. Then $[E, D] = E$ and by Lemma 8 we have

$$f(E)g(D) = g\left(D + E\frac{\partial}{\partial E}\ln f(E)\right) \cdot f(E)$$

for $f, g \in K[x][[h]]$. Since

$$f(E_2) = L_2^{-D_3} = (1 + E_2)^{-D_3} \quad \text{and} \quad E_2\frac{\partial}{\partial E_2}\ln f(E_2) = -E_2 L_2^{-1} D_3$$

the left-hand side of (10) may be written as

$$(L_1)^{-D_3} \cdot L_1^{-(D_2 - E_2 L_2^{-1} D_3)} \cdot L_2^{-D_3}.$$

Comparing this with the right-hand side of (10) we see that the exponents of L_1 are equal. □

8. Guessing deformation formulas

It remains mysterious how to guess the special form $P = (1 + hE_1)^{D_2}$ in Gerstenhaber's formula. The tight connection between the deformation of the algebra structure of A', the comultiplication of B', and the adjustment of the 2-cocycle S_h described in Theorem 10 allows to construct deformation formulas step by step, increasing the order of h taken into account.

Up to first order of h. i.e. (mod h^2) we have $P = 1 + hS$ and Eq. (4) is equivalent to the condition $\delta(S) = 0$. Thus there is a one-to-one correspondence between 2-cocycles of the coalgebra B and solutions P of (4) up to first order.

For the new comultiplication in B_h defined by Theorem 10 as

$$\Delta_h(b) = P^{-1} \cdot \Delta_B(b) \cdot P \equiv (1 - hS)\Delta_B(b)(1 + hS) \pmod{h^2}$$

we get $\Delta_h(b) \equiv \Delta_B(b) + h\dot{\Delta}(b) \pmod{h^2}$ with $\dot{\Delta}(b) := [\Delta_B(b), S]$ and for the new coboundary operator δ_h of B_h

$$\delta_h(S) \equiv \delta(S) - h\dot{\Delta}_1(S) + h\dot{\Delta}_2(S) \pmod{h^2}.$$

Hence the B-cocycle S may not be a B_h-cocycle. To prolongate the deformation to the next order S has to be changed to $S_h \equiv S + hS' \pmod{h^2}$ such that

$$\delta(S') = \dot{\Delta}_1(S) - \dot{\Delta}_2(S).$$

For Gerstenhaber's example this first order deformation generated by the 2-cocycle $S = E_1 D_2$ yields

$$\dot{\Delta}(E) = E_1[E_2, D_2] = E_1 E_2, \quad \dot{\Delta}(D) = [D_1, E_1] D_2 = -E_1 D_2$$

and hence

$$\Delta_h(E) \equiv E_1 + E_2 + hE_1E_2 = E_1 + (1 + hE_1)E_2 \pmod{h^2},$$

$$\Delta_h(D) \equiv D_1 + D_2 - hE_1D_2 = D_1 + (1 - hE_1)D_2 \pmod{h^2}$$

and

$$\delta_h(S) \equiv -2hE_1E_2D_3 \pmod{h^2}.$$

For the 2-cochain $E_1^2D_2 = E^2 \otimes D \in B \otimes B$ we get

$$\delta(E^2 \otimes D) = \delta(E^2) \otimes D = -2E_1E_2D_3.$$

Thus the B-cocycle S may be lifted (mod h^2) to the B_h-cocycle

$$S_h = E_1D_2 - hE_1^2D_2 = (1 - hE_1)E_1D_2.$$

This suggests to test whether

$$\Delta_h(E) = E_1 + L_1E_2, \quad \Delta_h(D) = D_1 + L_1^{-1}D_2$$

with $L := 1 + h\,E \in B_h$ describes the desired deformation of the comultiplication of B. Direct computations show that this is indeed the case and since $\Delta_h(L) = L_1L_2$ we can apply Theorem 11 to get the desired formula for P.

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